Uniqueness of the Infinite Open Cluster in Bernoulli Percolation

Columbia Directed Reading Program Spring '23

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What is Bernoulli percolation?

- Consider the infinite graph $G = (\mathbb{V}, \mathbb{E})$ such that $\mathbb{V} = \mathbb{Z}^d$, the d-dimensional integer lattice.
- Edges $\mathbb{E}$ exist between all vertically or horizontally adjacent vertices.
- Let each edge $e \in \mathbb{E}$ have a probability p of being 'open', independently of all other edges.

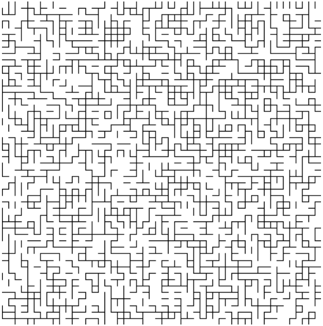
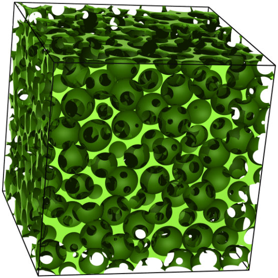


Figure: Bernoulli percolation with $p = 0.51$ for $d = 2$. Open edges are marked as black lines.

Motivations of Percolation Theory

Percolation Theory: Imagine immersing a large porous stone in a bucket of water. What is the probability of the center of the stone getting wet?



Percolation theory was inspired by the physical problem of modeling the flow of liquid through porous materials.
Also useful as a model for

- Forest fires
- Electrical networks
- Epidemics

We focus on the case of Bernoulli percolation on the square lattice since it is the most well-understood model.

Phase transition in percolations

Something happens as we increase p in the infinite square lattice...

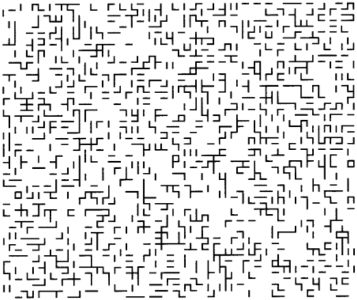


Figure: $p = 0.25$

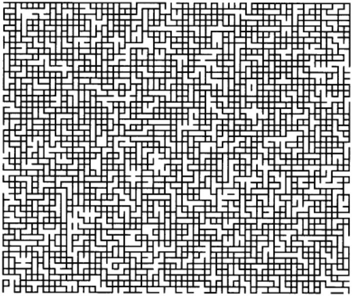


Figure: $p = 0.75$

As we increase the value of p, many of the open edges begin to join a single connected component of open edges.

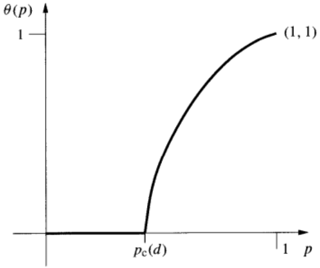
Phase transition in percolations

Definition

$\theta(p) := \mathbb{P}_p\,[0 \text{ is in an infinite open cluster }]$
(Probability of the origin being connected to an infinite open cluster on the square lattice percolation with parameter p)

Definition

$p_c := \sup\{p \in [0,1] : \theta(p) = 0\}$ (Critical point of the percolation)



square lattice: $p_c = \frac{1}{2}$
triangular lattice: $p_c = 2\sin(\frac{\pi}{18})$
hexagonal lattice: $p_c = 1 - 2\sin(\frac{\pi}{18})$

Uniqueness of the infinite open cluster

Theorem (Uniqueness of the infinite open cluster)

If $p \in [0,1]$ is s.t. $\theta(p) > 0$, then $\mathbb{P}_p[\exists \textit{ exactly one infinite cluster }] = 1$

At p just barely above the critical point, there will almost surely be *exactly* one infinite open cluster! (And just below the critical point, almost surely zero infinite clusters)

- To prove this result, we rely the property that the measure $\mathbb{P}_p$ we've been working with is **ergodic**, or that all events which are invariant under translation have probability either 0 or 1.
- We also rely on the **FKG Inequality** which states that for increasing events A and B, $\mathbb{P}_p[A \cap B] \geq \mathbb{P}_p[A]\mathbb{P}_p[B]$. Increasing events are those A such that opening more edges does not take you outside of A. Formally $\omega \in A \text{ and } \omega \leq \omega' \implies \omega' \in A$.

Observations about $\theta(p)$

Theorem (Existence of infinite open cluster)

The probability $\psi(p)$ that there exists an infinite open cluster satisfies

$$\psi(p) = \begin{cases} 0 & \text{if } \theta(p) = 0 \\ 1 & \text{if } \theta(p) > 0 \end{cases}$$

Proof.

Let $C(x)$ be the event that vertex x is in an infinite open cluster. Observe that this event is equally likely for any $x \in \mathbb{Z}^2$. Then we have that $\mathbb{P}_p[C(x)] = \theta(p)$.

- If $\theta(p) = 0$, $\psi(p) \leq \sum_{x \in \mathbb{Z}^2} \mathbb{P}_p[C(x)] = \sum_{x \in \mathbb{Z}^2} 0 = 0$.
- If $\theta(p) > 0$, then $\psi(p) \geq \mathbb{P}_p[C(0)] = \theta(p) > 0$, then $\psi(p) = 1$ by ergodicity (Hewitt-Savage zero-one law)

□

Doesn't say anything about the **number** of infinite clusters present.

Uniqueness of the infinite open cluster

Let $\mathcal{E}_{\leq 1}, \mathcal{E}_{<\infty}$, and $\mathcal{E}_{\infty}$ be the events that there are no more than one, finitely many, and infinitely many infinite clusters respectively.
Due to the existence theorem, it is sufficient to show that $\mathbb{P}_p[\mathcal{E}_{\leq 1}] = 1$ (the number of infinite clusters is either 0 or 1).
We first show that $\mathbb{P}_p[\mathcal{E}_{<\infty} \backslash \mathcal{E}_{\leq 1}] = 0$.

- By ergodicity, $\mathbb{P}_p[\mathcal{E}_{<\infty}]$ and $\mathbb{P}_p[\mathcal{E}_{\leq 1}]$ are 0 or 1.
- As $\mathcal{E}_{\leq 1} \subset \mathcal{E}_{<\infty}$, it suffices to show that $\mathbb{P}_p[\mathcal{E}_{<\infty}] > 0 \implies \mathbb{P}_p[\mathcal{E}_{\leq 1}] > 0$.

Uniqueness of the infinite open cluster

Want to Show: $\mathbb{P}_p[\mathcal{E}_{<\infty}] > 0 \implies \mathbb{P}_p[\mathcal{E}_{\leq 1}] > 0$

- Let $\mathcal{F}$ be the event that all (there may be none) the infinite clusters in the graph intersect with Λ_n, the boundary of the $n \times n$ square of centered at the origin. Denote the set of edges inside the $n \times n$ square as E_n.
- Consider the event $\mathcal{F} \cap \{\forall e \in E_n : \omega(e) = 1\}$. The open edges in E_n connect all of the infinite open clusters so that this event is equivalent to $\mathcal{E}_{\leq 1}$.
- By the FKG Inequality, $\mathbb{P}_p[\mathcal{F} \cap \{\forall e \in E_n : \omega(e) = 1\}] \geq \mathbb{P}_p[\mathcal{F}] \cdot \left(\prod_{e \in E_n} \mathbb{P}_p[\omega(e) = 1]\right) = \mathbb{P}_p[\mathcal{F}]p^{|E_n|}$
- Let us assume that $\mathbb{P}_p[\mathcal{E}_{<\infty}] > 0$. We can then choose n sufficiently large such that $\mathbb{P}_p[\mathcal{F}] \geq \frac{1}{2}\mathbb{P}_p[\mathcal{E}_{<\infty}] > 0$, hence $\mathbb{P}_p[\mathcal{E}_{\leq 1}] = \mathbb{P}_p[\mathcal{F} \cap \{\forall e \in E_n : \omega(e) = 1\}] \geq \mathbb{P}_p[\mathcal{F}]p^{|E_n|} > 0$.
- This completes the proof that $\mathbb{P}_p[\mathcal{E}_{<\infty} \backslash \mathcal{E}_{\leq 1}] = 0$. (That the number of infinite clusters will be either 0, 1, or ∞).

Uniqueness of the infinite open cluster

Next, we rule out the possibility of infinitely many infinite clusters by showing that $\mathbb{P}_p[\mathcal{E}_\infty] = 0$.
Consider n large enough that $\mathbb{P}_p[K \text{ infinite clusters intersect } \Lambda_n] \geq \frac{1}{2}\mathbb{P}_p[\mathcal{E}_\infty]$ with K large enough that at least three vertices x, y, z of Λ_n at a distance at least 3 from one another are all connected to infinity. We can modify the configuration inside E_n to create a 'trifurcation'.

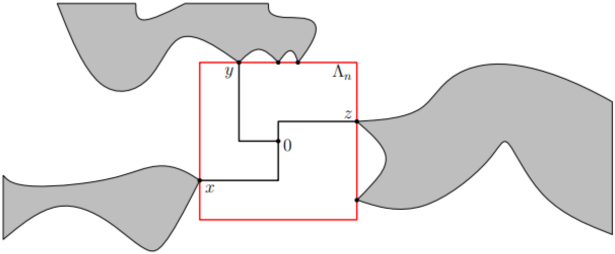


Figure: A trifurcation

Uniqueness of the infinite open cluster

- A trifurcation is a point where three open paths connected to three distinct infinite open clusters intersect at a single common point.
- Let $\mathcal{T}_x$ be the event that x is a trifurcation. We can deduce that $\mathbb{P}_p[\mathcal{T}_0] \geq [p(1-p)]^{|E_n|} \cdot \frac{1}{2}\mathbb{P}_p[\mathcal{E}_\infty]$
- T, the number of trifurcations in E_n, has expected value $\mathbb{E}_p[T] = \mathbb{P}_p[\mathcal{T}_0] \cdot |E_n|$
- Since each trifurcation corresponds to three distinct points on the boundary, we can deduce that the number of trifurcations inside E_n is upper bounded by the length of the outer boundary: $T \leq |\Lambda_n|$.
- Then, $\mathbb{P}_p[\mathcal{T}_0] = \frac{\mathbb{E}_p[T]}{|E_n|} \leq \frac{|\Lambda_n|}{|E_n|} \to 0$ as $n \to \infty$ due to volume growing at a faster rate than surface area.
- This combined with $\mathbb{P}_p[\mathcal{E}_\infty] \leq \frac{2\mathbb{P}_p[\mathcal{T}_0]}{[p(1-p)]^{|E_n|}}$ implies that $\mathbb{P}_p[\mathcal{E}_\infty] = 0$.

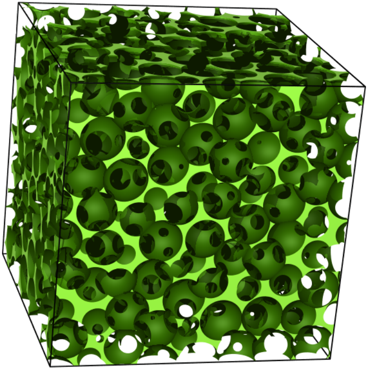
Therefore, $\mathbb{P}_p[\mathcal{E}_{\leq 1}] = 1$. □

Conclusion

In conclusion, we have shown that

Theorem (Uniqueness of the infinite open cluster)

If $p \in [0, 1]$ *is s.t.* $\theta(p) > 0$, *then* $\mathbb{P}_p[\exists \textit{ exactly one infinite cluster }] = 1$



Bonus: Identifying the critical point p_c for $d = 2$

- Define the dual lattice of $\mathbb{Z}^2$ to be $(\mathbb{Z}^2)^* := (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$. Observe that each edge e^* of the dual crosses a corresponding edge e.
- For an edge configuration $\omega = (\omega(e) : e \in \mathbb{E}) \in \{0,1\}^{|\mathbb{E}|}$ on $\mathbb{Z}^2$, there is a dual configuration ω^* on $(\mathbb{Z}^2)^*$ s.t. $\forall e \in \mathbb{E} : \omega^*(e^*) = 1 - \omega(e)$.
- **Heuristic:** If an edge is labeled 1 with prob. p, its dual edge is labeled 1 with prob. $1 - p$. As we increase p from 0 to 1, the infinite cluster should appear on $\mathbb{Z}^2$ at the same p that it disappears from $(\mathbb{Z}^2)^*$. Hence $p_c = 1 - p_c \implies p_c = \frac{1}{2}$.

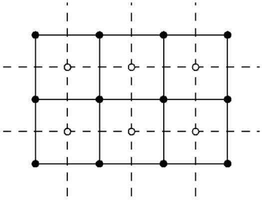


Figure: $\mathbb{Z}^2$ and its dual $(\mathbb{Z}^2)^*$